Stresses and Deformation in a Circular Matrix Subject to Internal Pressure Gradients

JOAB J. BLECH*
Ford Motor Company, Dearborn, Mich.

Nomenclature

r = radial coordinate
σ_r, σ_θ = radial and tangential normal stresses
p = pressure
u = radial displacement
$\epsilon_r, \epsilon_\theta$ = radial and tangential linear strains
E_r, E_θ = radial and tangential Young's moduli
$\nu_{r\theta}, \nu_{\theta r}$ = Poisson's ratios
γ^2 = E_θ/E_r = anisotropy parameter
L = characteristic length
p_o = characteristic pressure
ρ = r/L = nondimensional radius
S_r = σ_r/p_o,
S_θ = σ_θ/p_o = nondimensional radial and tangential stresses, respectively
P = p/p_o = nondimensional pressure
U = u/L = nondimensional radial displacement
w, v = nondimensional radii
a, b = disk inner and outer radii
α = a/b = inner to outer radius ratio
λ, μ = tangential and radial pressure reduction factors

Introduction

MODERN engineering structures include matrices that consist of an ensemble, for example, of many plate facets. One such structure is a heat exchanger. Other similar structures may be encountered whenever weight reduction is of importance, and the ratio of load carrying capacity to the structure weight is to be as large as possible.

There are two alternatives in attacking the stress analysis problem in such a matrix. The "microscopic" approach where the geometry of the individual matrix cells, the external loads, and their elastic properties are defined. An analysis will then yield the stresses in each individual web of the structure. There are several major disadvantages to this approach: 1) the geometry definition of the matrix may be a laborious and practically impossible task, 2) the measurement of web properties could be difficult, and 3) a large scale computer and an enormous simultaneous equations solver routine may be necessary. The second approach is "macroscopic," where the matrix is looked upon as an elastic continuum, probably anisotropic. The disadvantages that exist in the microscopic approach are eliminated now. Geometry definition merely reduces to the description of the body boundaries. Properties are measured on a gross scale, and the use of computers may, sometimes, be completely eliminated by the derivation of closed form solutions or, in other cases, a moderately sized computer and simultaneous equations solver will suffice.

We concern ourselves here with the axisymmetric plane stress problems of such a matrix, when subjected to internal pressure gradients, a common thing to occur in rotary heat exchangers. The structure is assumed to possess a cylindrical anisotropy on a macroscopic scale.

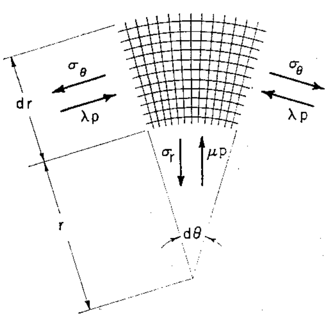

Fig. 1 Matrix element.

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* Turbine Research Specialist.

Analytical Formulation

Consider a matrix element subject to stresses and pressure loads as shown in Fig. 1. Force equilibrium considerations lead then to the equation:

$$(d/dr)[r(\sigma_r - \mu p)] - \sigma_\theta + \lambda p = 0 \tag{1}$$

The pressure loads in the radial and tangential directions are multiplied by reduction factors. Those factors represent the percentage open area that is exposed to the pressure. Assuming the matrix behaves as a cylindrical aelotropic body, we have the following stress strain relations:

$$\epsilon_r = (\sigma_r - \nu_{r\theta}\sigma_\theta)/E_r, \tag{2a}$$

$$\epsilon_\theta = (\sigma_\theta - \nu_{\theta r}\tau_r)/E_\theta \tag{2b}$$

together with the reciprocity relation:

$$E_\theta \nu_{r\theta} = E_r \nu_{\theta r} \tag{3}$$

The strain compatibility equation is

$$(d/dr)(r\epsilon_\theta) = \epsilon_r \tag{4}$$

Eliminating σ_θ from Eqs. (2a) and (2b) by means of Eq. (1) and inserting the resulting equations into Eq. (4) leads to

$$r^2 \frac{d^2\sigma_r}{dr^2} + 3r\frac{d\sigma_r}{dr} + (1-\gamma^2)\sigma_r = \nu_{\theta r}(\lambda - \mu)p + (2 + \nu_{\theta r})\mu_r \frac{dp}{dr} + \mu r^2 \frac{d^2p}{dr^2} \tag{5}$$

Introduce the nondimensional variables

$$\rho = r/L;\ S_r = \sigma_r/p_o;\ S_\theta = \sigma_\theta/p_o;\ P = p/p_o;\ U = u/L \tag{6}$$

Equation (5) becomes, after nondimensionalizing and condensing terms

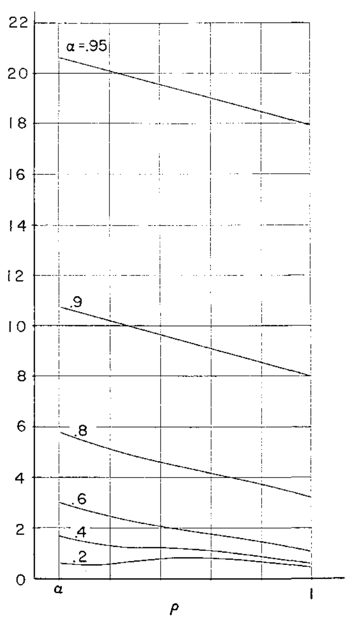

Fig. 2 Tangential stresses.

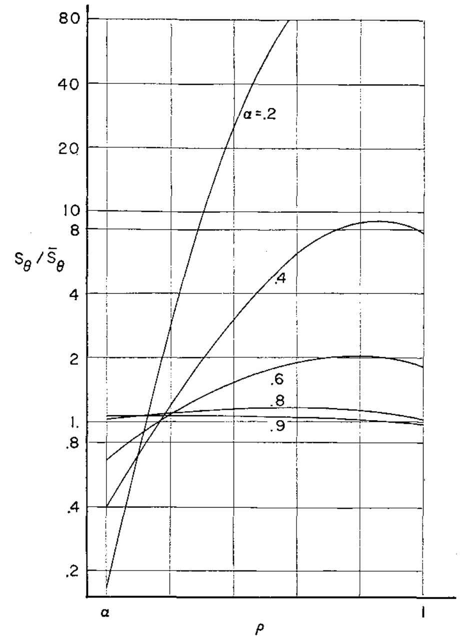

Fig. 3 Tangential stress comparison.

$$\rho^{\gamma}\frac{d}{d\rho}\left[\rho^{\,1-2\gamma}\frac{d}{d\rho}(\rho^{1+\gamma}S_r)\right] = \nu_{\theta r}(\lambda-\mu)P + (2 + \nu_{\theta r})\,\mu\rho\frac{dP}{d\rho} + \mu\rho^2\frac{d^2P}{d\rho^2} \qquad (7)$$

Equation (7) yields, after two integrations

$$S_r = C_1/\rho^{1-\gamma} + C_2/\rho^{1+\gamma} + \mu P - \mu(1-\nu_{\theta r})/\rho^{1+\gamma} \times \int^{\rho} v^{\gamma}P(v)dv + [\nu_{\theta r}(\lambda-\mu)(1-\gamma)(\gamma + \nu_{\theta r})]/\rho^{1+\gamma}\int^{\rho} v^{2+\gamma}\int^{v} w^{-\gamma}P(w)dwdv, \qquad (8)$$

where C_1 and C_2 are integration constants.

The tangential stress and the radial displacement are then found from the expressions

$$S_\theta = \lambda P + (d/d\rho)\,[\rho(\sigma_r - \mu P)], \qquad (9a)$$

$$U = \rho(S_\theta - \nu_{\theta r}S_r)p_0/E_\theta \qquad (9b)$$

Numerical Example

Consider a circular disc subjected to a pressure p_0 at its inner radius a, which drops linearly to zero at the outer radius, b. Let $L = b$ and $\alpha = a/b$, and assume $\lambda = \mu = 1$. The pressure is then given by

$$P = (1-\rho)/(1-\alpha) \qquad (10)$$

Inserting Eq. (10) into Eqs. (8–9b) yields for the radial and tangential stresses and the radial deformation:

$$S_r = (2 + \nu_{\theta r})\rho/[(\gamma^2 - 4)(1-\alpha)] + C_1/\rho^{1-\gamma} + C_2/\rho^{1+\gamma} \qquad (11a)$$

$$S_\theta = (\gamma^2 + 2\nu_{\theta r})\,\rho/[(\gamma^2-4)(1-\alpha)] + \gamma C_1/\rho^{1-\gamma} - \gamma C_2/\rho^{1-\gamma} \qquad (11b)$$

$$U = \{(\gamma^2 - \nu_{\theta r}{}^2)\rho^2/[(\gamma^2-4)(1-\alpha)] + (\gamma - \nu_{\theta r})C_1\rho^{\gamma} - (\gamma+\nu_{\theta r})C_2\rho^{-\gamma}\}p_o/E_\theta, \qquad (11c)$$

provided $\gamma \neq 2$. If the disk is assumed to be traction free at its inner and outer radius, C_1 and C_2 will yield

$$C_1 = -(2 + \nu_{\epsilon r})(\alpha^{\gamma+2} - 1)/[(\gamma^2-4)(1-\alpha)(\alpha^{2\gamma}-1)] \qquad (12a)$$

$$C_2 = (2 + \nu_{\theta r})(\alpha^{\gamma+2} - \alpha^{2\gamma})/[(\gamma-4)(1-\alpha)\,(\alpha^{2\gamma}-1)] \qquad (12b)$$

The tangential stresses are plotted for the case $\gamma^2 = 16$ and $\nu_{\theta r} = 1.8$ in Fig. 2 vs ρ, for various values of α. Those values of γ^2 and $\nu_{\theta r}$ are typical for a PYROCERAM glass ceramic matrix structure. When the disk is subjected to an internal pressure p_o only, the tangential stresses are

$$\bar{S}_\theta = \gamma\alpha^{1+\gamma}(1/\rho^{1+\gamma} + 1/\rho^{1-\gamma})/(1-\alpha^{2\gamma}) \qquad (13)$$

In Fig. 3, the ratio $S_\theta/\bar{S}_\theta$ is plotted vs ρ, for various values of α. It is seen that S_θ varies considerably from $\bar{S}_\theta$ except in the neighborhood of $\alpha = 1$. In the latter case S_θ and $\bar{S}_\theta$ may be approximated by:

$$S_\theta = (1+\alpha)/[2(1-\alpha)];\ \bar{S}_\theta = \alpha(1-\alpha)^{-1} \qquad (14)$$

These expressions are obtained from equilibrium considerations of the disk element in Fig. 1 when dr is replaced by $b - a$ and for r, the average radius $(b + a)/2$ is used.

Heat Transfer in Separated Laminar Hypersonic Flow

J. W. Hodgson*
The University of Tennessee, Knoxville, Tenn.

Nomenclature

c_p = constant pressure specific heat (assumed constant)
L = reattachment zone length
M_e = local freestream Mach number
n = summing index
r = recovery factor ($Pr^{1/2}$ for laminar flow)
Re_L = U_dL/ν_c
Re_w = U_ew/ν_e
S^* = Denison and Baum[6] parameter
S_b = surface distance from leading stagnation point of body to leading edge of cavity
St_y = local Stanton number, $q/[\rho_c v c_p(T_d - T_w)]$
St_L = Stanton number at $y = L$
T_{aw} = freestream laminar adiabatic wall temperature
T_e = local freestream temperature
T_w = cavity wall temperature
U_e = local freestream velocity
v = local velocity outside boundary layer on cavity wall
γ = ratio of specific heats (c_p/c_v)
ρ_c = density in cavity
ν_c = kinematic viscosity in cavity
ν_e = local freestream kinematic viscosity

The flow of a fluid over and within a cavity formed in a surface has been the subject of much research.[1–9] From these studies it is clear that the floor of the cavity will, in general, experience a lower convective heat flux than would a smooth surface exposed to the same external flow conditions. Hunt and Howell,[1] pursuing means for the thermal protection of hypersonic vehicles, found that adding an open honeycomb structure to a surface significantly decreased the temperature of that surface in hypersonic flow—provided the cell geometry was such that the cell depth was more than twice the cell width. Because most of the previously reported data were obtained for shallow (width greater than depth) cavities, Wieting[2] measured the heat flux distribution in single deep cavities and found good agreement between his data and Burggraf's[3] theory. There remained a need, however, for a

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* Associate Professor, Mechanical and Aerospace Engineering Department.